

# Extremal Unimodular Lattices in Dimension 36

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Dedicated to Professor Vladimir D. Tonchev on His 60th Birthday

## Abstract

In this paper, new extremal odd unimodular lattices in dimension 36 are constructed. Some new odd unimodular lattices in dimension 36 with long shadows are also constructed.

## 1 Introduction

A (Euclidean) lattice  $L \subset \mathbb{R}^n$  in dimension  $n$  is *unimodular* if  $L = L^*$ , where the dual lattice  $L^*$  of  $L$  is defined as  $\{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$  under the standard inner product  $(x, y)$ . A unimodular lattice is called *even* if the norm  $(x, x)$  of every vector  $x$  is even. A unimodular lattice, which is not even, is called *odd*. An even unimodular lattice in dimension  $n$  exists if and only if  $n \equiv 0 \pmod{8}$ , while an odd unimodular lattice exists for every dimension. Two lattices  $L$  and  $L'$  are *isomorphic*, denoted  $L \cong L'$ , if there exists an orthogonal matrix  $A$  with  $L' = L \cdot A$ , where  $L \cdot A = \{xA \mid x \in L\}$ . The automorphism group  $\text{Aut}(L)$  of  $L$  is the group of all orthogonal matrices  $A$  with  $L = L \cdot A$ .

Rains and Sloane [17] showed that the minimum norm  $\min(L)$  of a unimodular lattice  $L$  in dimension  $n$  is bounded by  $\min(L) \leq 2\lfloor n/24 \rfloor + 2$  unless

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$n = 23$  when  $\min(L) \leq 3$ . We say that a unimodular lattice meeting the upper bound is *extremal*.

The smallest dimension for which there is an odd unimodular lattice with minimum norm (at least) 4 is 32 (see [13]). There are exactly five odd unimodular lattices in dimension 32 having minimum norm 4, up to isomorphism [4]. For dimensions 33, 34 and 35, the minimum norm of an odd unimodular lattice is at most 3 (see [13]). The next dimension for which there is an odd unimodular lattice with minimum norm (at least) 4 is 36. Four extremal odd unimodular lattices in dimension 36 are known, namely,  $\text{Sp}_4(4)\text{D8.4}$  in [13],  $G_{36}$  in [6, Table 2],  $N_{36}$  in [7, Section 3] and  $A_4(C_{36})$  in [8, Section 3]. Recently, one more lattice has been found, namely,  $A_6(C_{36,6}(D_{18}))$  in [9, Table II]. This situation motivates us to improve the number of known non-isomorphic extremal odd unimodular lattices in dimension 36. The main aim of this paper is to prove the following:

**Proposition 1.** *There are at least 26 non-isomorphic extremal odd unimodular lattices in dimension 36.*

The above proposition is established by constructing new extremal odd unimodular lattices in dimension 36 from self-dual  $\mathbb{Z}_k$ -codes, where  $\mathbb{Z}_k$  is the ring of integers modulo  $k$ , by using two approaches. One approach is to consider self-dual  $\mathbb{Z}_4$ -codes. Let  $B$  be a binary doubly even code of length 36 satisfying the following conditions:

$$\text{the minimum weight of } B \text{ is at least } 16, \quad (1)$$

$$\text{the minimum weight of its dual code } B^\perp \text{ is at least } 4. \quad (2)$$

Then a self-dual  $\mathbb{Z}_4$ -code with residue code  $B$  gives an extremal odd unimodular lattice in dimension 36 by Construction A. We show that a binary doubly even  $[36, 7]$  code satisfying the conditions (1) and (2) has weight enumerator  $1 + 63y^{16} + 63y^{20} + y^{36}$  (Lemma 2). It was shown in [15] that there are four codes having the weight enumerator, up to equivalence. We construct ten new extremal odd unimodular lattices in dimension 36 from self-dual  $\mathbb{Z}_4$ -codes whose residue codes are doubly even  $[36, 7]$  codes satisfying the conditions (1) and (2) (Lemma 4). New odd unimodular lattices in dimension 36 with minimum norm 3 having shadows of minimum norm 5 are constructed from some of the new lattices (Proposition 7). These are often called unimodular lattices with long shadows (see [14]). The other

approach is to consider self-dual  $\mathbb{Z}_k$ -codes ( $k = 5, 6, 7, 9, 19$ ), which have generator matrices of a special form given in (7). Eleven more new extremal odd unimodular lattices in dimension 36 are constructed by Construction A (Lemma 8). Finally, we give a certain short observation on ternary self-dual codes related to extremal odd unimodular lattices in dimension 36.

All computer calculations in this paper were done by MAGMA [1].

## 2 Preliminaries

### 2.1 Unimodular lattices

Let  $L$  be an odd unimodular lattice and let  $L_0$  denote the even sublattice, that is, the sublattice of vectors of even norms. Then  $L_0$  is a sublattice of  $L$  of index 2 [4]. The *shadow*  $S(L)$  of  $L$  is defined to be  $L_0^* \setminus L$ . There are cosets  $L_1, L_2, L_3$  of  $L_0$  such that  $L_0^* = L_0 \cup L_1 \cup L_2 \cup L_3$ , where  $L = L_0 \cup L_2$  and  $S = L_1 \cup L_3$ . Shadows for odd unimodular lattices appeared in [4] and also in [5, p. 440], in order to provide restrictions on the theta series of odd unimodular lattices. Two lattices  $L$  and  $L'$  are *neighbors* if both lattices contain a sublattice of index 2 in common. If  $L$  is an odd unimodular lattice in dimension divisible by 4, then there are two unimodular lattices containing  $L_0$ , which are rather than  $L$ , namely,  $L_0 \cup L_1$  and  $L_0 \cup L_3$ . Throughout this paper, we denote the two unimodular neighbors by

$$Ne_1(L) = L_0 \cup L_1 \text{ and } Ne_2(L) = L_0 \cup L_3. \quad (3)$$

The theta series  $\theta_L(q)$  of  $L$  is the formal power series  $\theta_L(q) = \sum_{x \in L} q^{(x,x)}$ . The kissing number of  $L$  is the second nonzero coefficient of the theta series of  $L$ , that is, the number of vectors of minimum norm in  $L$ . Conway and Sloane [4] gave some characterization of theta series of odd unimodular lattices and their shadows. Using [4, (2), (3)], it is easy to determine the possible theta series  $\theta_{L_{36}}(q)$  and  $\theta_{S(L_{36})}(q)$  of an extremal odd unimodular lattice  $L_{36}$  in dimension 36 and its shadow  $S(L_{36})$ :

$$\theta_{L_{36}}(q) = 1 + (42840 + 4096\alpha)q^4 + (1916928 - 98304\alpha)q^5 + \cdots, \quad (4)$$

$$\theta_{S(L_{36})}(q) = \alpha q + (960 - 60\alpha)q^3 + (3799296 + 1734\alpha)q^5 + \cdots, \quad (5)$$

respectively, where  $\alpha$  is a nonnegative integer. It follows from the coefficients of  $q$  and  $q^3$  in  $\theta_{S(L_{36})}(q)$  that  $0 \leq \alpha \leq 16$ .

## 2.2 Self-dual $\mathbb{Z}_k$ -codes and Construction A

Let  $\mathbb{Z}_k$  be the ring of integers modulo  $k$ , where  $k$  is a positive integer greater than 1. A  $\mathbb{Z}_k$ -code  $C$  of length  $n$  is a  $\mathbb{Z}_k$ -submodule of  $\mathbb{Z}_k^n$ . Two  $\mathbb{Z}_k$ -codes are *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. A code  $C$  is *self-dual* if  $C = C^\perp$ , where the dual code  $C^\perp$  of  $C$  is defined as  $\{x \in \mathbb{Z}_k^n \mid x \cdot y = 0 \text{ for all } y \in C\}$ , under the standard inner product  $x \cdot y$ .

If  $C$  is a self-dual  $\mathbb{Z}_k$ -code of length  $n$ , then the following lattice

$$A_k(C) = \frac{1}{\sqrt{k}} \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid (x_1 \bmod k, \dots, x_n \bmod k) \in C\}$$

is a unimodular lattice in dimension  $n$ . This construction of lattices is called Construction A.

## 3 From self-dual $\mathbb{Z}_4$ -codes

From now on, we omit the term odd for odd unimodular lattices in dimension 36, since all unimodular lattices in dimension 36 are odd. In this section, we construct ten new non-isomorphic extremal unimodular lattices in dimension 36 from self-dual  $\mathbb{Z}_4$ -codes by Construction A. Five new non-isomorphic unimodular lattices in dimension 36 with minimum norm 3 having shadows of minimum norm 5 are also constructed.

### 3.1 Extremal unimodular lattices

Every  $\mathbb{Z}_4$ -code  $C$  of length  $n$  has two binary codes  $C^{(1)}$  and  $C^{(2)}$  associated with  $C$ :

$$C^{(1)} = \{c \bmod 2 \mid c \in C\} \text{ and } C^{(2)} = \{c \bmod 2 \mid c \in \mathbb{Z}_4^n, 2c \in C\}.$$

The binary codes  $C^{(1)}$  and  $C^{(2)}$  are called the residue and torsion codes of  $C$ , respectively. If  $C$  is a self-dual  $\mathbb{Z}_4$ -code, then  $C^{(1)}$  is a binary doubly even code with  $C^{(2)} = C^{(1)\perp}$  [3]. Conversely, starting from a given binary doubly even code  $B$ , a method for construction of all self-dual  $\mathbb{Z}_4$ -codes  $C$  with  $C^{(1)} = B$  was given in [16, Section 3].

The Euclidean weight of a codeword  $x = (x_1, \dots, x_n)$  of  $C$  is  $m_1(x) + 4m_2(x) + m_3(x)$ , where  $m_\alpha(x)$  denotes the number of components  $i$  with

$x_i = \alpha$  ( $\alpha = 1, 2, 3$ ). The minimum Euclidean weight  $d_E(C)$  of  $C$  is the smallest Euclidean weight among all nonzero codewords of  $C$ . It is easy to see that  $\min\{d(C^{(1)}), 4d(C^{(2)})\} \leq d_E(C)$ , where  $d(C^{(i)})$  denotes the minimum weight of  $C^{(i)}$  ( $i = 1, 2$ ). In addition,  $d_E(C) \leq 4d(C^{(2)})$  and  $A_4(C)$  has minimum norm  $\min\{4, d_E(C)/4\}$  (see e.g. [7]). Hence, if there is a binary doubly even code  $B$  of length 36 satisfying the conditions (1) and (2), then an extremal unimodular lattice in dimension 36 is constructed as  $A_4(C)$ , through a self-dual  $\mathbb{Z}_4$ -code  $C$  with  $C^{(1)} = B$ . If there is a binary  $[36, k]$  code  $B$  satisfying the conditions (1) and (2), then  $k = 7$  or 8 (see [2]).

**Lemma 2.** *Let  $B$  be a binary doubly even  $[36, 7]$  code satisfying the conditions (1) and (2). Then the weight enumerator of  $B$  is  $1 + 63y^{16} + 63y^{20} + y^{36}$ .*

*Proof.* The weight enumerator of  $B$  is written as:

$$W_B(y) = 1 + ay^{16} + by^{20} + cy^{24} + dy^{28} + ey^{32} + (2^7 - 1 - a - b - c - d - e)y^{36},$$

where  $a, b, c, d$  and  $e$  are nonnegative integers. By the MacWilliams identity, the weight enumerator of  $B^\perp$  is given by:

$$\begin{aligned} W_{B^\perp}(y) = & 1 + \frac{1}{16}(-567 + 5a + 4b + 3c + 2d + e)y \\ & + \frac{1}{2}(1260 - 10a - 10b - 9c - 7d - 4e)y^2 \\ & + \frac{1}{16}(-112455 + 885a + 900b + 883c + 770d + 497e)y^3 + \dots \end{aligned}$$

Since  $d(B^\perp) \geq 4$ , the weight enumerator of  $B$  is written using  $a$  and  $b$ :

$$\begin{aligned} W_B(y) = & 1 + ay^{16} + by^{20} + (882 - 10a - 4b)y^{24} + (-1638 + 20a + 6b)y^{28} \\ & + (1197 - 15a - 4b)y^{32} + (-314 + 4a + b)y^{36}. \end{aligned}$$

Suppose that  $B$  does not contain the all-one vector  $\mathbf{1}$ . Then  $b = 314 - 4a$ . In this case, since the coefficients of  $y^{24}$  and  $y^{28}$  are  $-374 + 6a$  and  $246 - 4a$ , these yield that  $a \geq 62$  and  $a \leq 61$ , respectively, which gives the contradiction. Hence,  $B$  contains  $\mathbf{1}$ . Then  $b = 315 - 4a$ . Since the coefficient  $a - 63$  of  $y^{32}$  is 0, the weight enumerator of  $B$  is uniquely determined as  $1 + 63y^{16} + 63y^{20} + y^{36}$ .  $\square$

*Remark 3.* A similar approach shows that the weight enumerator of a binary doubly even  $[36, 8]$  code  $B$  satisfying the conditions (1) and (2) is uniquely determined as  $1 + 153y^{16} + 72y^{20} + 30y^{24}$ .

It was shown in [15] that there are four inequivalent binary  $[36, 7, 16]$  codes containing **1**. The four codes are doubly even. Hence, there are exactly four binary doubly even  $[36, 7]$  codes satisfying the conditions (1) and (2), up to equivalence. The four codes are optimal in the sense that these codes achieve the Gray–Rankin bound, and the codewords of weight 16 are corresponding to quasi-symmetric SDP  $2$ -(36, 16, 12) designs [11]. Let  $B_{36,i}$  be the binary doubly even  $[36, 7, 16]$  code corresponding to the quasi-symmetric SDP  $2$ -(36, 16, 12) design, which is the residual design of the symmetric SDP  $2$ -(64, 28, 12) design  $D_i$  in [15, Section 5] ( $i = 1, 2, 3, 4$ ). As described above, all self-dual  $\mathbb{Z}_4$ -codes  $C$  with  $C^{(1)} = B_{36,i}$  have  $d_E(C) = 16$  ( $i = 1, 2, 3, 4$ ). Hence,  $A_4(C)$  are extremal.

Using the method in [16, Section 3], self-dual  $\mathbb{Z}_4$ -codes  $C$  are constructed from  $B_{36,i}$ . Then ten extremal unimodular lattices  $A_4(C_{36,i})$  ( $i = 1, 2, \dots, 10$ ) are constructed, where  $C_{36,i}^{(1)} = B_{36,2}$  ( $i = 1, 2, 3$ ),  $C_{36,i}^{(1)} = B_{36,3}$  ( $i = 4, 5, 6, 7$ ) and  $C_{36,i}^{(1)} = B_{36,4}$  ( $i = 8, 9, 10$ ). To distinguish between the known lattices and our lattices, we give in Table 1 the kissing numbers  $\tau(L)$ ,  $\{n_1(L), n_2(L)\}$  and the orders  $\# \text{Aut}(L)$  of the automorphism groups, where  $n_i(L)$  denotes the number of vectors of norm 3 in  $Ne_i(L)$  defined in (3) ( $i = 1, 2$ ). These have been calculated by MAGMA. Table 1 shows the following:

**Lemma 4.** *The five known lattices and the ten extremal unimodular lattices  $A_4(C_{36,i})$  ( $i = 1, 2, \dots, 10$ ) are non-isomorphic to each other.*

*Remark 5.* In this way, we have found two more extremal unimodular lattices  $A_4(C)$ , where  $C$  are self-dual  $\mathbb{Z}_4$ -codes with  $C^{(1)} = B_{36,1}$ . However, we have verified by MAGMA that the two lattices are isomorphic to  $N_{36}$  in [7] and  $A_4(C_{36})$  in [8].

*Remark 6.* For  $L = A_4(C_{36,i})$  ( $i = 1, 2, \dots, 10$ ), it follows from  $\tau(L)$  and  $\{n_1(L), n_2(L)\}$  that one of the two unimodular neighbors  $Ne_1(L)$  and  $Ne_2(L)$  defined in (3) is extremal. We have verified by MAGMA that the extremal one is isomorphic to  $A_4(C_{36,i})$ .

For  $i = 1, 2, \dots, 10$ , the code  $C_{36,i}$  is equivalent to some code  $\overline{C_{36,i}}$  with generator matrix of the form:

$$\begin{pmatrix} I_7 & A & B_1 + 2B_2 \\ O & 2I_{22} & 2D \end{pmatrix}, \quad (6)$$

where  $A, B_1, B_2, D$  are  $(1, 0)$ -matrices,  $I_n$  denotes the identity matrix of order  $n$  and  $O$  denotes the  $22 \times 7$  zero matrix. We only list in Figure 1 the  $7 \times 29$

Table 1: Extremal unimodular lattices in dimension 36

Lattices $L$	$\tau(L)$	$\{n_1(L), n_2(L)\}$	$\# \text{Aut}(L)$
$\text{Sp}_4(4)\text{D8.4}$ in [13]	42840	$\{480, 480\}$	31334400
$G_{36}$ in [6, Table 2]	42840	$\{144, 816\}$	576
$N_{36}$ in [7]	42840	$\{0, 960\}$	849346560
$A_4(C_{36})$ in [8]	51032	$\{0, 840\}$	660602880
$A_6(C_{36,6}(D_{18}))$ in [9]	42840	$\{384, 576\}$	288
$A_4(C_{36,1})$ in Section 3	51032	$\{0, 840\}$	6291456
$A_4(C_{36,2})$ in Section 3	42840	$\{0, 960\}$	6291456
$A_4(C_{36,3})$ in Section 3	51032	$\{0, 840\}$	22020096
$A_4(C_{36,4})$ in Section 3	51032	$\{0, 840\}$	1966080
$A_4(C_{36,5})$ in Section 3	51032	$\{0, 840\}$	1572864
$A_4(C_{36,6})$ in Section 3	42840	$\{0, 960\}$	2621440
$A_4(C_{36,7})$ in Section 3	42840	$\{0, 960\}$	1966080
$A_4(C_{36,8})$ in Section 3	42840	$\{0, 960\}$	393216
$A_4(C_{36,9})$ in Section 3	51032	$\{0, 840\}$	1376256
$A_4(C_{36,10})$ in Section 3	51032	$\{0, 840\}$	393216
$A_5(D_{36,1})$ in Section 4	42840	$\{144, 816\}$	144
$A_5(D_{36,2})$ in Section 4	42840	$\{456, 504\}$	72
$A_6(D_{36,3})$ in Section 4	42840	$\{240, 720\}$	288
$A_6(D_{36,4})$ in Section 4	42840	$\{240, 720\}$	576
$A_7(D_{36,5})$ in Section 4	42840	$\{288, 672\}$	288
$A_7(D_{36,6})$ in Section 4	42840	$\{144, 816\}$	72
$A_7(D_{36,7})$ in Section 4	42840	$\{144, 816\}$	288
$A_9(D_{36,8})$ in Section 4	42840	$\{384, 576\}$	144
$A_{19}(D_{36,9})$ in Section 4	42840	$\{288, 672\}$	144
$A_5(E_{36,1})$ in Section 4	42840	$\{456, 504\}$	72
$A_6(E_{36,2})$ in Section 4	42840	$\{384, 576\}$	144

matrix  $M_i = \begin{pmatrix} A & B_1 + 2B_2 \end{pmatrix}$  to save space. Note that  $\begin{pmatrix} O & 2I_{22} & 2D \end{pmatrix}$  in (6) can be obtained from  $\begin{pmatrix} I_7 & A & B_1 + 2B_2 \end{pmatrix}$  since  $\overline{C_{36,i}}^{(2)} = \overline{C_{36,i}}^{(1)\perp}$ . A generator matrix of  $A_4(C_{36,i})$  is obtained from that of  $C_{36,i}$ .

### 3.2 Unimodular lattices with long shadows

The possible theta series of a unimodular lattice  $L$  in dimension 36 having minimum norm 3 and its shadow are as follows:

$$\begin{aligned} &1 + (960 - \alpha)q^3 + (42840 + 4096\beta)q^4 + \cdots, \\ &\beta q + (\alpha - 60\beta)q^3 + (3833856 - 36\alpha + 1734\beta)q^5 + \cdots, \end{aligned}$$

respectively, where  $\alpha$  and  $\beta$  are integers with  $0 \leq \beta \leq \frac{\alpha}{60} < 16$  [7]. Then the kissing number of  $L$  is at most 960 and  $\min(S(L)) \leq 5$ . Unimodular lattices  $L$  with  $\min(L) = 3$  and  $\min(S(L)) = 5$  are often called unimodular lattices with long shadows (see [14]). Only one unimodular lattice  $L$  in dimension 36 with  $\min(L) = 3$  and  $\min(S(L)) = 5$  was known, namely,  $A_4(C_{36})$  in [7].

Let  $L$  be one of  $A_4(C_{36,2})$ ,  $A_4(C_{36,6})$ ,  $A_4(C_{36,7})$  and  $A_4(C_{36,8})$ . Since  $\{n_1(L), n_2(L)\} = \{0, 960\}$ , one of the two unimodular neighbors  $Ne_1(L)$  and  $Ne_2(L)$  in (3) is extremal and the other is a unimodular lattice  $L'$  with minimum norm 3 having shadow of minimum norm 5. We denote such lattices  $L'$  constructed from  $A_4(C_{36,2})$ ,  $A_4(C_{36,6})$ ,  $A_4(C_{36,7})$  and  $A_4(C_{36,8})$  by  $N_{36,1}$ ,  $N_{36,2}$ ,  $N_{36,3}$  and  $N_{36,4}$ , respectively. We list in Table 2 the orders  $\# \text{Aut}(N_{36,i})$  ( $i = 1, 2, 3, 4$ ) of the automorphism groups, which have been calculated by MAGMA. Table 2 shows the following:

**Proposition 7.** *There are at least 5 non-isomorphic unimodular lattices  $L$  in dimension 36 with  $\min(L) = 3$  and  $\min(S(L)) = 5$ .*

## 4 From self-dual $\mathbb{Z}_k$ -codes ( $k \geq 5$ )

In this section, we construct more extremal unimodular lattices in dimension 36 from self-dual  $\mathbb{Z}_k$ -codes ( $k \geq 5$ ).

Let  $A^T$  denote the transpose of a matrix  $A$ . An  $n \times n$  matrix is negacir-



$$\begin{aligned}
M_1 &= \begin{pmatrix} 01101111111000000110000333322 \\ 10101101001000111110011210223 \\ 11000011001110100111100010321 \\ 11010100101011010011012032011 \\ 00110101010101001101013312302 \\ 0000000000000111111112111333 \\ 00011111111110000001111213133 \end{pmatrix} & M_2 &= \begin{pmatrix} 01101111111000000110002111100 \\ 10101101001000111110011210023 \\ 11000011001110100111100010123 \\ 11010100101011010011012032211 \\ 00110101010101001101013312300 \\ 00000000000001111111121113131 \\ 00011111111110000001111211331 \end{pmatrix} \\
M_3 &= \begin{pmatrix} 01101111111000000110002131300 \\ 10101101001000111110011210223 \\ 11000011001110100111100010121 \\ 11010100101011010011012032011 \\ 00110101010101001101013310102 \\ 0000000000000111111112131131 \\ 00011111111110000001111233331 \end{pmatrix} & M_4 &= \begin{pmatrix} 01101111111001111101100220313 \\ 10101101001001001110101231032 \\ 11000011001110110100102101101 \\ 00111001100011100101113010223 \\ 11011000011101100011100021110 \\ 0000000000000111111112111131 \\ 00011111111110000001113301311 \end{pmatrix} \\
M_5 &= \begin{pmatrix} 01101111111001111101102202133 \\ 10101101001001001110101231032 \\ 11000011001110110100102101101 \\ 00111001100011100101113010223 \\ 11011000011101100011100221310 \\ 0000000000000111111112111131 \\ 00011111111110000001113301311 \end{pmatrix} & M_6 &= \begin{pmatrix} 01101111111001111101102000313 \\ 10101101001001001110101231032 \\ 11000011001110110100102101103 \\ 00111001100011100101113210223 \\ 11011000011101100011100021110 \\ 0000000000000111111112313113 \\ 00011111111110000001113103333 \end{pmatrix} \\
M_7 &= \begin{pmatrix} 01101111111001111101102202313 \\ 10101101001001001110101231030 \\ 11000011001110110100102101121 \\ 00111001100011100101113210201 \\ 11011000011101100011100021112 \\ 0000000000000111111112111113 \\ 00011111111110000001113301333 \end{pmatrix} & M_8 &= \begin{pmatrix} 1111111000011111111000022313 \\ 11110001100111000000001130313 \\ 01000110111001110100011231021 \\ 10011100001011100001101233011 \\ 01010101010010101101003301321 \\ 0000000001111111111112331111 \\ 01111111100000000111111233313 \end{pmatrix} \\
M_9 &= \begin{pmatrix} 1111111000011111111002200111 \\ 11110001100111000000001332311 \\ 01000110111001110100013211021 \\ 10011100001011100001103213011 \\ 01010101010010101101000331321 \\ 01111111100000000111111211113 \\ 000000000111111111111303313 \end{pmatrix} & M_{10} &= \begin{pmatrix} 1111111000011111111002202313 \\ 11110001100111000000001130311 \\ 01000110111001110100011211021 \\ 10011100001011100001101213011 \\ 01010101010010101101003321123 \\ 0000000001111111111112313313 \\ 01111111100000000111111211111 \end{pmatrix}
\end{aligned}$$

Figure 1: Generator matrices of  $\overline{C_{36,i}}$  ( $i = 1, 2, \dots, 10$ )

Table 2:  $\# \text{Aut}(N_{36,i})$  ( $i = 1, 2, 3, 4$ )

Lattices $L$	$\# \text{Aut}(L)$
$A_4(C_{36})$ in [7]	1698693120
$N_{36,1}$	12582912
$N_{36,2}$	5242880
$N_{36,3}$	3932160
$N_{36,4}$	786432

culant if it has the following form:

$$\begin{pmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ -r_{n-1} & r_0 & \cdots & r_{n-2} \\ -r_{n-2} & -r_{n-1} & \cdots & r_{n-3} \\ \vdots & \vdots & & \vdots \\ -r_1 & -r_2 & \cdots & r_0 \end{pmatrix}.$$

Let  $D_{36,i}$  ( $i = 1, 2, \dots, 9$ ) and  $E_{36,i}$  ( $i = 1, 2$ ) be  $\mathbb{Z}_k$ -codes of length 36 with generator matrices of the following form:

$$\begin{pmatrix} I_{18} & A & B \\ & -B^T & A^T \end{pmatrix}, \quad (7)$$

where  $k$  are listed in Table 3,  $A$  and  $B$  are  $9 \times 9$  negacirculant matrices with first rows  $r_A$  and  $r_B$  listed in Table 3. It is easy to see that these codes are self-dual since  $AA^T + BB^T = -I_9$ . Thus,  $A_k(D_{36,i})$  ( $i = 1, 2, \dots, 9$ ) and  $A_k(E_{36,i})$  ( $i = 1, 2$ ) are unimodular lattices, for  $k$  given in Table 3. In addition, we have verified by MAGMA that these lattices are extremal.

To distinguish between the above eleven lattices and the known 15 lattices, in Table 1 we give  $\tau(L)$ ,  $\{n_1(L), n_2(L)\}$  and  $\# \text{Aut}(L)$ , which have been calculated by MAGMA. The two lattices have the identical  $(\tau(L), \{n_1(L), n_2(L)\}, \# \text{Aut}(L))$  for each of the pairs  $(A_5(E_{36,1}), A_5(D_{36,2}))$  and  $(A_6(E_{36,2}), A_9(D_{36,8}))$ . However, we have verified by MAGMA that the two lattices are non-isomorphic for each pair. Therefore, we have the following:

**Lemma 8.** *The 26 lattices in Table 1 are non-isomorphic to each other.*

Lemma 8 establishes Proposition 1.

Table 3: Self-dual  $\mathbb{Z}_k$ -codes of length 36

Codes	$k$	$r_A$	$r_B$
$D_{36,1}$	5	(0, 0, 0, 1, 2, 2, 0, 4, 2)	(0, 0, 0, 0, 4, 3, 3, 0, 1)
$D_{36,2}$	5	(0, 0, 0, 1, 3, 0, 2, 0, 4)	(3, 0, 4, 1, 4, 0, 0, 4, 4)
$D_{36,3}$	6	(0, 1, 5, 3, 2, 0, 3, 5, 1)	(3, 1, 0, 0, 5, 1, 1, 1, 1)
$D_{36,4}$	6	(0, 1, 3, 5, 1, 5, 5, 4, 4)	(4, 0, 3, 2, 4, 5, 5, 2, 4)
$D_{36,5}$	7	(0, 1, 6, 3, 5, 0, 4, 5, 4)	(0, 1, 6, 3, 5, 2, 1, 5, 1)
$D_{36,6}$	7	(0, 1, 1, 3, 2, 6, 1, 4, 6)	(0, 1, 4, 0, 5, 3, 6, 2, 0)
$D_{36,7}$	7	(0, 0, 0, 1, 5, 5, 5, 1, 1)	(0, 5, 4, 2, 5, 1, 1, 3, 6)
$D_{36,8}$	9	(0, 0, 0, 1, 0, 4, 3, 0, 0)	(0, 4, 1, 5, 3, 5, 1, 7, 0)
$D_{36,9}$	19	(0, 0, 0, 1, 15, 15, 9, 6, 5)	(14, 16, 0, 14, 15, 8, 8, 3, 12)
$E_{36,1}$	5	(0, 1, 0, 2, 1, 3, 2, 2, 0)	(2, 0, 1, 0, 1, 1, 2, 3, 1)
$E_{36,2}$	6	(0, 1, 5, 3, 4, 4, 1, 1, 0)	(4, 0, 1, 3, 4, 2, 3, 0, 1)

*Remark 9.* Similar to Remark 6, it is known [7] that the extremal neighbor is isomorphic to  $L$  for the case where  $L$  is  $N_{36}$  in [7], and we have verified by MAGMA that the extremal neighbor is isomorphic to  $L$  for the case where  $L$  is  $A_4(C_{36})$  in [8].

## 5 Related ternary self-dual codes

In this section, we give a certain short observation on ternary self-dual codes related to some extremal odd unimodular lattices in dimension 36.

### 5.1 Unimodular lattices from ternary self-dual codes

Let  $T_{36}$  be a ternary self-dual code of length 36. The two unimodular neighbors  $Ne_1(A_3(T_{36}))$  and  $Ne_2(A_3(T_{36}))$  given in (3) are described in [10] as  $L_S(T_{36})$  and  $L_T(T_{36})$ . In this section, we use the notation  $L_S(T_{36})$  and  $L_T(T_{36})$ , instead of  $Ne_1(A_3(T_{36}))$  and  $Ne_2(A_3(T_{36}))$ , since the explicit constructions and some properties of  $L_S(T_{36})$  and  $L_T(T_{36})$  are given in [10]. By Theorem 6 in [10] (see also Theorem 3.1 in [6]),  $L_T(T_{36})$  is extremal when  $T_{36}$  satisfies the following condition (a), and both  $L_S(T_{36})$  and  $L_T(T_{36})$  are extremal when  $T_{36}$  satisfies the following condition (b):

- (a) extremal (minimum weight 12) and admissible (the number of 1's in the components of every codeword of weight 36 is even),
- (b) minimum weight 9 and maximum weight 33.

For each of (a) and (b), no ternary self-dual code satisfying the condition is currently known.

## 5.2 Condition (a)

Suppose that  $T_{36}$  satisfies the condition (a). Since  $T_{36}$  has minimum weight 12,  $A_3(T_{36})$  has minimum norm 3 and kissing number 72. By Theorem 6 in [10],  $\min(L_T(T_{36})) = 4$  and  $\min(L_S(T_{36})) = 3$ . Hence, since the shadow of  $L_T(T_{36})$  contains no vector of norm 1, by (4) and (5)  $L_T(T_{36})$  has theta series  $1 + 42840q^4 + 1916928q^5 + \dots$ . It follows that  $\{n_1(L_T(T_{36})), n_2(L_T(T_{36}))\} = \{72, 888\}$ .

By Theorem 1 in [12], the possible complete weight enumerator  $W_C(x, y, z)$  of a ternary extremal self-dual code  $C$  of length 36 containing  $\mathbf{1}$  is written as

$$a_1\delta_{36} + a_2\alpha_{12}^3 + a_3\alpha_{12}^2\beta_6^2 + a_4\alpha_{12}(\beta_6^2)^2 + a_5(\beta_6^2)^3 + a_6\beta_6\gamma_{18}\alpha_{12} + a_7\beta_6\gamma_{18}\beta_6^2,$$

using some  $a_i \in \mathbb{R}$  ( $i = 1, 2, \dots, 7$ ), where  $\alpha_{12} = a(a^3 + 8p^3)$ ,  $\beta_6 = a^2 - 12b$ ,  $\gamma_{18} = a^6 - 20a^3p^3 - 8p^6$ ,  $\delta_{36} = p^3(a^3 - p^3)^3$  and  $a = x^3 + y^3 + z^3$ ,  $p = 3xyz$ ,  $b = x^3y^3 + x^3z^3 + y^3z^3$ . From the minimum weight, we have the following:

$$\begin{aligned} a_2 &= \frac{3281}{13824} - \frac{a_1}{64}, a_3 = \frac{203}{4608} - \frac{9a_1}{256}, a_4 = \frac{1763}{13824} + \frac{3a_1}{128}, \\ a_5 &= -\frac{277}{13824} - \frac{a_1}{256}, a_6 = \frac{1133}{1728} + \frac{3a_1}{64}, a_7 = -\frac{77}{1728} - \frac{a_1}{64}. \end{aligned}$$

Since  $W_C(x, y, z)$  contains the term  $(15180 + 2916a_1)y^{15}z^{21}$ , if  $C$  is admissible, then

$$a_1 = -\frac{15180}{2916}.$$

Hence, the complete weight enumerator of a ternary admissible extremal self-dual code containing  $\mathbf{1}$  is uniquely determined, which is listed in Figure 2.

$$\begin{aligned}
& x^{36} + y^{36} + z^{36} + 78706260x^{12}y^{12}z^{12} \\
& + 682(x^{18}y^{18} + x^{18}z^{18} + y^{18}z^{18}) + 7019232(x^{15}y^{15}z^6 + x^{15}y^6z^{15} + x^6y^{15}z^{15}) \\
& + 29172(x^{24}y^6z^6 + x^6y^{24}z^6 + x^6y^6z^{24}) + 10260316(x^{18}y^9z^9 + x^9y^{18}z^9 + x^9y^9z^{18}) \\
& + 37995408(x^{12}y^{15}z^9 + x^{12}y^9z^{15} + x^{15}y^{12}z^9 + x^{15}y^9z^{12} + x^9y^{12}z^{15} + x^9y^{15}z^{12}) \\
& + 3924756(x^{12}y^{18}z^6 + x^{12}y^6z^{18} + x^{18}y^{12}z^6 + x^{18}y^6z^{12} + x^6y^{12}z^{18} + x^6y^{18}z^{12}) \\
& + 58344(x^{12}y^{21}z^3 + x^{12}y^3z^{21} + x^{21}y^{12}z^3 + x^{21}y^3z^{12} + x^3y^{12}z^{21} + x^3y^{21}z^{12}) \\
& + 102(x^{12}y^{24} + x^{12}z^{24} + x^{24}y^{12} + x^{24}z^{12} + y^{12}z^{24} + y^{24}z^{12}) \\
& + 170544(x^{15}y^{18}z^3 + x^{15}y^3z^{18} + x^{18}y^{15}z^3 + x^{18}y^3z^{15} + x^3y^{15}z^{18} + x^3y^{18}z^{15}) \\
& + 641784(x^{21}y^6z^9 + x^{21}y^9z^6 + x^6y^{21}z^9 + x^6y^9z^{21} + x^9y^{21}z^6 + x^9y^6z^{21}) \\
& + 6732(x^{24}y^3z^9 + x^{24}y^9z^3 + x^3y^{24}z^9 + x^3y^9z^{24} + x^9y^{24}z^3 + x^9y^3z^{24})
\end{aligned}$$

Figure 2: Complete weight enumerator

### 5.3 Condition (b)

Suppose that  $T_{36}$  satisfies the condition (b). By the Gleason theorem (see Corollary 5 in [12]), the weight enumerator of  $T_{36}$  is uniquely determined as:

$$\begin{aligned}
& 1 + 888y^9 + 34848y^{12} + 1432224y^{15} + 18377688y^{18} + 90482256y^{21} \\
& + 162551592y^{24} + 97883072y^{27} + 16178688y^{30} + 479232y^{33}.
\end{aligned}$$

By Theorem 6 in [10] (see also Theorem 3.1 in [6]),  $L_S(T_{36})$  and  $L_T(T_{36})$  are extremal. Hence,  $\min(A_3(T_{36})) = 3$  and  $\min(S(A_3(T_{36}))) = 5$ .

Note that a unimodular lattice  $L$  contains a 3-frame if and only if  $L \cong A_3(C)$  for some ternary self-dual code  $C$ . Let  $L_{36}$  be any of the five lattices given in Table 2. Let  $L_{36}^{(3)}$  be the set  $\{\{x, -x\} \mid (x, x) = 3, x \in L_{36}\}$ . We define the simple undirected graph  $\Gamma(L_{36})$ , whose set of vertices is the set of 480 pairs in  $L_{36}^{(3)}$  and two vertices  $\{x, -x\}, \{y, -y\} \in L_{36}^{(3)}$  are adjacent if  $(x, y) = 0$ . It follows that the 3-frames in  $L_{36}$  are precisely the 36-cliques in the graph  $\Gamma(L_{36})$ . We have verified by MAGMA that  $\Gamma(L_{36})$  are regular graphs with valency 368, and the maximum sizes of cliques in  $\Gamma(L_{36})$  are 12. Hence, none of these lattices is constructed from some ternary self-dual code by Construction A.

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